

1- اثبات المتساوية الأولى

ليكن x عنصرا من $\mathbb{R}$

$$\left(\frac{2x+1}{\sqrt{3}}\right)^2 = \frac{4x^2 + 4x + 1}{3} \quad \text{لدينا}$$

$$\frac{3}{4} \left(1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2\right) = \frac{3}{4} \left(\frac{4x^2 + 4x + 4}{3}\right) \quad \text{ومنه}$$

$$= x^2 + x + 1$$

$$x^2 + x + 1 = \frac{3}{4} \left[1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2\right] \quad \text{إذن}$$

ب- حساب التكامل I

$$dt = \frac{2}{\sqrt{3}} dx \quad \text{ويكون لدينا} \quad t = \frac{2x+1}{\sqrt{3}} \quad \text{نضع}$$

$$dx = \frac{\sqrt{3}}{2} dt \quad \text{أي}$$

$$t = \sqrt{3} \quad \text{أي} \quad t = \frac{3}{\sqrt{3}} \quad \text{إذا كان } x = 1 \quad \text{فإن}$$

$$t = 1 \quad \text{فإن} \quad x = \frac{\sqrt{3}-1}{2} \quad \text{إذا كان}$$

ومنه

$$I = \int_1^{\sqrt{3}} \frac{1}{\frac{3}{4}(1+t^2)} \cdot \frac{\sqrt{3}}{2} dt$$

$$= \frac{2\sqrt{3}}{3} \int_1^{\sqrt{3}} \frac{1}{1+t^2} dt$$

$$= \frac{2\sqrt{3}}{3} [\text{Arctan } t]_1^{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3} \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{2\sqrt{3}}{3} \cdot \frac{\pi}{12}$$

2- أ- التحقق من المتساوية الثانية

ليكن x من $\mathbb{R}^*$

$$\frac{1}{x} - \frac{2x+1}{2(x^2+x+1)} = \frac{2x^2+2x+2-2x^2-x}{2x(x^2+x+1)} \quad \text{لدينا}$$

$$= \frac{x+2}{2x(x^2+x+1)}$$

ومنه

$$\left| \frac{1}{x} - \frac{2x+1}{2(x^2+x+1)} - \frac{1}{2(x^2+x+1)} = \frac{x+2}{2x(x^2+x+1)} - \frac{x}{2x(x^2+x+1)} \right|$$

$$= \frac{1}{x(x^2+x+1)}$$

$$\frac{1}{x(x^2+x+1)} = \frac{1}{x} - \frac{2x+1}{2(x^2+x+1)} - \frac{1}{2(x^2+x+1)} \quad \text{إذن}$$

ب- حساب التكامل J

لدينا

$$J = \int_{\frac{\sqrt{3}-1}{2}}^1 \left[\frac{1}{x} - \frac{2x+1}{2(x^2+x+1)} - \frac{1}{2(x^2+x+1)} \right] dx$$

$$= \int_{\frac{\sqrt{3}-1}{2}}^1 \frac{1}{x} dx - \frac{1}{2} \int_{\frac{\sqrt{3}-1}{2}}^1 \frac{2x+1}{(x^2+x+1)} dx - \frac{1}{2} \int_{\frac{\sqrt{3}-1}{2}}^1 \frac{1}{x^2+x+1} dx$$

$$= [\ln x]_{\frac{\sqrt{3}-1}{2}}^1 - \frac{1}{2} [\ln(x^2+x+1)]_{\frac{\sqrt{3}-1}{2}}^1 - \frac{1}{2} I$$

$$= -\ln \frac{\sqrt{3}-1}{2} - \frac{1}{2} \left(\ln 3 - \ln \frac{3}{2} \right) - \frac{1}{2} \cdot \frac{2\sqrt{3}}{3} \cdot \frac{\pi}{12}$$

$$= -\ln \frac{\sqrt{3}-1}{2} - \frac{\ln 2}{2} - \frac{\sqrt{3}}{3} \cdot \frac{\pi}{12}$$

3- حساب التكامل K

لدينا

$$K = \int_{\frac{\sqrt{3}-1}{2}}^1 \frac{2x+1}{(x^2+x+1)^2} \ln x dx$$

$$= \int_{\frac{\sqrt{3}-1}{2}}^1 u'(x) \ln x dx$$

$$\forall x \in \mathbb{R} \quad u(x) = \frac{-1}{x^2 + x + 1}$$

حيث

$$K = [u(x) \ln x]_{\frac{\sqrt{3}-1}{2}}^1 - \int_{\frac{\sqrt{3}-1}{2}}^1 u(x) \ln' x \, dx$$

ومنه

$$= [u(x) \ln x]_{\frac{\sqrt{3}-1}{2}}^1 + \int_{\frac{\sqrt{3}-1}{2}}^1 \frac{1}{x(x^2 + x + 1)} \, dx$$

$$= -\ln \frac{\sqrt{3}-1}{2} u\left(\frac{\sqrt{3}-1}{2}\right) + J$$

$$= +\frac{2}{3} \ln \frac{\sqrt{3}-1}{2} + \left(-\ln \frac{\sqrt{3}-1}{2} - \frac{1}{2} \ln 2 - \frac{\sqrt{3}}{3} \cdot \frac{\pi}{12}\right)$$

$$= -\frac{1}{3} \ln \frac{\sqrt{3}-1}{2} - \frac{1}{2} \ln 2 - \frac{\sqrt{3}}{3} \cdot \frac{\pi}{12}$$

