

$$\begin{cases} u_0 = 3 \\ u_{n+1} = 2 + \frac{1}{u_n} - \frac{2}{u_n^2} \end{cases} ; \quad n \in \mathbb{N}$$

- (1) لدينا $u_0 = 3$ إذن $u_0 > 2$.
- نفترض أن $u_p > 2$ لنبين أن $u_{p+1} > 2$.

لدينا

$$\begin{aligned} u_{p+1} - 2 &= \frac{1}{u_p} - \frac{2}{u_p^2} \\ &= \frac{u_p - 2}{u_p^2} \end{aligned}$$

$$u_p > 2 \Rightarrow \begin{cases} u_p - 2 > 0 \\ u_p^2 > 0 \end{cases}$$

$$\Rightarrow \frac{u_p - 2}{u_p^2} > 0$$

$$\Rightarrow u_{p+1} - 2 > 0$$

$$\Rightarrow u_{p+1} > 2$$

$$\boxed{\forall n \in \mathbb{N} ; 2 < u_n}$$

إذن

وبالتالي فإن المتتالية (u_n) مصغرة بالعدد 2.

$$(2 - x)(x^2 - 1) = 2x^2 - 2 - x^3 + x \quad (2) \text{ أ.}$$

$$\boxed{\forall x \in \mathbb{R} ; (2 - x)(x^2 - 1) = -x^3 + 2x^2 + x - 2} \quad \text{إذن :}$$

$$u_{n+1} - u_n = 2 + \frac{1}{u_n} - \frac{2}{u_n^2} - u_n \quad \text{ب. لدينا :}$$

$$= \frac{2u_n^2 + u_n - 2 - u_n^3}{u_n^2}$$

$$= \frac{(2 - u_n)(u_n^2 - 1)}{u_n^2}$$

$$2 < u_n \Rightarrow \begin{cases} 2 - u_n < 0 \\ u_n^2 - 1 > 0 \end{cases}$$

$$\Rightarrow \frac{(2 - u_n)(u_n^2 - 1)}{u_n^2} < 0$$

$$\Rightarrow u_{n+1} - u_n < 0$$

$$\boxed{\forall n \in \mathbb{N} ; u_{n+1} < u_n} \quad \text{إذن}$$

ومنه فإن المتتالية (u_n) تناقصية.

ج - بما أن المتتالية (u_n) تناقصية ومصفورة بالعدد 2 فإنها متقاربة.

$$(3) \quad \text{أ - لدينا} \quad u_{n+1} - 2 = \frac{u_n - 2}{u_n^2}$$

$$u_n > 2 \Rightarrow u_n^2 \geq 4$$

$$\Rightarrow \frac{1}{u_n^2} \leq \frac{1}{4}$$

$$\Rightarrow \frac{u_n - 2}{u_n^2} \leq \frac{1}{4} (u_n - 2)$$

$$\Rightarrow u_{n+1} - 2 \leq \frac{1}{4} (u_n - 2)$$

$$\boxed{\forall n \in \mathbb{N} ; u_{n+1} - 2 \leq \frac{1}{4} (u_n - 2)} \quad \text{إذن}$$

$$0 < u_1 - 2 \leq \frac{1}{4} (u_0 - 2)$$

$$\times \quad 0 < u_2 - 2 \leq \frac{1}{4} (u_1 - 2)$$

$$\times \quad \vdots$$

$$0 < u_n - 2 \leq \frac{1}{4} (u_{n-1} - 2)$$

$$0 < (u_1 - 2) \dots (u_n - 2) \leq \left(\frac{1}{4}\right)^n (u_0 - 2) (u_1 - 2) \dots (u_{n-1} - 2)$$

$$\Leftrightarrow 0 < u_n - 2 \leq \left(\frac{1}{4}\right)^n (u_0 - 2)$$

$$\boxed{\forall n \in \mathbb{N} ; u_n - 2 \leq \left(\frac{1}{4}\right)^n} \quad \text{إذن}$$

$$(4) \quad \text{بما أن} \quad -1 < \frac{1}{4} < 1 \quad \text{فإن} \quad \lim_{n \rightarrow +\infty} \left(\frac{1}{4}\right)^n = 0$$

$$0 < u_n - 2 \leq \left(\frac{1}{4}\right)^n \quad \text{وبما أن}$$

$$\lim_{n \rightarrow +\infty} u_n = 2 \quad \text{فإن}$$

Achamel