

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \frac{1}{8} (1 + \sqrt[3]{u_n})^3 \quad ; n \in \mathbb{N} \end{cases}$$

$$u_1 = \frac{1}{8} (1 + \sqrt[3]{u_0})^3 = \frac{1}{8} \quad (1)$$

$$u_2 = \frac{1}{8} (1 + \sqrt[3]{u_1})^3 = \frac{1}{8} \left(\frac{3}{2} \right)^3 = \frac{27}{64}$$

$$\begin{aligned} \forall n \in \mathbb{N} \quad ; \quad 0 \leq u_n < 1 & \quad \text{لنبين أن :} \\ 0 \leq u_0 < 1 \quad \text{إذن} \quad u_0 = 0 & \quad \bullet \text{ لدينا أن} \end{aligned}$$

$$\begin{aligned} (p \in \mathbb{N}) \quad 0 \leq u_p < 1 & \quad \bullet \text{ نفترض أن} \\ 0 \leq u_{p+1} < 1 & \quad \bullet \text{ لنبين أن} \end{aligned}$$

$$\begin{aligned} 0 \leq u_p < 1 & \Rightarrow 0 \leq \sqrt[3]{u_p} < 1 \\ & \Rightarrow 1 \leq 1 + \sqrt[3]{u_p} < 2 \\ & \Rightarrow 1 \leq (1 + \sqrt[3]{u_p})^3 < 8 \\ & \Rightarrow \frac{1}{8} \leq \frac{1}{8} (1 + \sqrt[3]{u_p})^3 < 1 \\ & \Rightarrow 0 \leq u_{p+1} < 1 \quad \left(0 \leq \frac{1}{8} \quad \text{لأن} \right) \end{aligned}$$

$$\forall n \in \mathbb{N} ; 0 \leq u_n < 1$$

إذن

$$u_{n+1} - u_n = \frac{1}{8} (1 + \sqrt[3]{u_n})^3 - u_n \quad (3)$$

$$\begin{aligned} &= \left[\frac{1}{2} (1 + \sqrt[3]{u_n}) \right]^3 - (\sqrt[3]{u_n})^3 \\ &= \left[\frac{1}{2} (1 + \sqrt[3]{u_n}) - \sqrt[3]{u_n} \right] \left[\frac{1}{4} (1 + \sqrt[3]{u_n})^2 + \frac{1}{2} \sqrt[3]{u_n} (1 + \sqrt[3]{u_n}) + (\sqrt[3]{u_n})^2 \right] \\ &= \frac{1}{2} (1 - \sqrt[3]{u_n}) \left[\frac{1}{4} (1 + \sqrt[3]{u_n})^2 + \frac{1}{2} \sqrt[3]{u_n} (1 + \sqrt[3]{u_n}) + (\sqrt[3]{u_n})^2 \right] \geq 0 \end{aligned}$$

(لأن $0 \leq u_n < 1$)

$$\forall n \in \mathbb{N} ; u_{n+1} \geq u_n$$

إذن

ومنه فإن (u_n) متتالية تزايدية.

تذكير :

$$a^3 - b^3 = (a - b) (a^2 + ab + b^2)$$

(4) بما أن المتتالية (u_n) تزايدية ومكبورة بالعدد 1 فإنها متقاربة.

(5)

$$\begin{aligned} a_{n+1} &= \sqrt[3]{u_{n+1}} - 1 \\ &= \sqrt[3]{\frac{1}{8} (1 + \sqrt[3]{u_n})^3} - 1 \\ &= \frac{1}{2} (1 + \sqrt[3]{u_n}) - 1 \\ &= \frac{1}{2} (\sqrt[3]{u_n} - 1) \\ &= \frac{1}{2} a_n \end{aligned}$$

$$\forall n \in \mathbb{N} ; a_{n+1} = \frac{1}{2} a_n$$

إذن

ومنه فإن المتتالية (a_n) هندسية أساسها $\frac{1}{2}$.

$$a_0 = \sqrt[3]{u_0} - 1 = -1$$

ب. * لدينا

$$\forall n \in \mathbb{N} ; a_n = -\left(\frac{1}{2}\right)^n$$

إذن

$$a_n = \sqrt[3]{u_n} - 1$$

وبما أن

$$u_n = (1 + a_n)^3$$

فإن

$$\forall n \in \mathbb{N} ; u_n = \left[1 - \left(\frac{1}{2}\right)^n\right]^3$$

إذن

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{2}\right)^n = 0 \quad \text{فإن} \quad -1 < \frac{1}{2} < 1 \quad * \text{ بما أن}$$

ومنه فإن

$$\lim_{n \rightarrow +\infty} u_n = 1$$

ج. *

$$\begin{aligned} S_n &= \sqrt[3]{u_0} + \sqrt[3]{u_1} + \sqrt[3]{u_2} + \dots + \sqrt[3]{u_n} \\ &= (1 + a_0) + (1 + a_1) + (1 + a_2) + \dots + (1 + a_n) \\ &= \underbrace{(1 + 1 + \dots + 1)}_{(n+1)} + (a_0 + a_1 + \dots + a_n) \end{aligned}$$

$$= (n+1) + \frac{a_0 \left(1 - \left(\frac{1}{2}\right)^{n+1}\right)}{1 - \frac{1}{2}}$$

$$= (n+1) - 2 \left[1 - \left(\frac{1}{2}\right)^{n+1}\right]$$

$$\forall n \in \mathbb{N} ; S_n = n - 1 + \left(\frac{1}{2}\right)^n$$

إذن

$$\left. \begin{aligned} \lim_{n \rightarrow +\infty} \left(\frac{1}{2}\right)^n &= 0 \\ \lim_{n \rightarrow +\infty} (n - 1) &= +\infty \end{aligned} \right\} \Rightarrow \lim_{n \rightarrow +\infty} S_n = +\infty \quad *$$