

$$u_n = 2 \int_0^1 \frac{x^{2n+1}}{1+x^2} dx$$

$$\frac{x^{2(n+1)+1}}{1+x^2} - \frac{x^{2n+1}}{1+x^2} = \frac{x^{2n+1}(x^2-1)}{1+x^2} \quad \text{لدينا : } (1)$$

$$0 \leq x \leq 1 \Rightarrow x^{2n+1} \geq 0 \quad , \quad x^2 - 1 \leq 0 \quad , \quad 1 + x^2 > 0$$

$$\Rightarrow \frac{x^{2n+1}(x^2-1)}{1+x^2} \leq 0$$

$$\Rightarrow 2 \int_0^1 \frac{x^{2n+1}(x^2-1)}{1+x^2} dx \leq 0$$

$$\Rightarrow 2 \int_0^1 \frac{x^{2(n+1)+1}}{1+x^2} dx - 2 \int_0^1 \frac{x^{2n+1}}{1+x^2} dx \leq 0$$

$$\boxed{\forall n \in \mathbb{N} \quad u_{n+1} - u_n \leq 0} \quad \text{إذن}$$

لدينا :

$$\frac{x^{2(n+1)+1}}{1+x^2} + \frac{x^{2n+1}}{1+x^2} = \frac{x^{2n+1}(x^2+1)}{x^2+1} = x^{2n+1}$$

$$u_{n+1} + u_n = 2 \int_0^1 \frac{x^{2(n+1)+1} + x^{2n+1}}{1+x^2} dx \quad \text{إذن}$$

$$= 2 \int_0^1 x^{2n+1} dx$$

$$= \left[2 \frac{1}{2(n+1)} x^{2n+2} \right]_0^1 = \left[\frac{1}{n+1} x^{2n+2} \right]_0^1$$

$$\boxed{\forall n \in \mathbb{N} \quad u_{n+1} + u_n = \frac{1}{n+1}} \quad \text{إذن}$$

$$\forall x \in [0, 1] \quad \frac{x^{2n+1}}{1+x^2} \geq 0$$

$$\Rightarrow 2 \int_0^1 \frac{x^{2n+1}}{1+x^2} dx \geq 0$$

$$\boxed{\forall n \in \mathbb{N} \quad u_n \geq 0}$$

إذن :

$$u_0 = 2 \int_0^1 \frac{x}{1+x^2} dx \quad * (2)$$

$$= [\ln(1+x^2)]_0^1$$

$$\boxed{u_0 = \ln 2}$$

إذن :

$$u_1 + u_0 = \frac{1}{1} \quad \text{لدينا :}$$

$$\Rightarrow \boxed{u_1 = 1 - u_0 = 1 - \ln 2}$$

$$u_2 + u_1 = \frac{1}{2} \quad \text{لدينا :}$$

$$\Rightarrow \boxed{u_2 = \frac{1}{2} - u_1 = \ln 2 - \frac{1}{2}}$$

$$(3) \text{ بما أن } u_{n+1} \geq 0 \text{ و } u_n \geq 0 \text{ و } u_{n+1} + u_n = \frac{1}{n+1} \text{ فإن}$$

$$u_n \leq \frac{1}{n+1}$$

$$\forall n \in \mathbb{N} \quad 0 \leq u_n \leq \frac{1}{n+1} \quad \text{إذن :}$$

$$(4) \text{ بما أن } 0 \leq u_n \leq \frac{1}{n+1} \text{ لكل } n \text{ من } \mathbb{N}$$

$$\text{و } \lim_{n \rightarrow +\infty} u_n = 0 \text{ و } (u_n) \text{ متقاربة فإن } \lim_{n \rightarrow +\infty} \left(\frac{1}{n+1} \right) = 0$$

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